



Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra

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Abstract: We recall and improve our 2019 concepts of *n-Power Set of a Set*, *n-SuperHyperGraph*, *Plithogenic n-SuperHyperGraph*, and *n-ary HyperAlgebra*, *n-ary NeutroHyperAlgebra*, *n-ary AntiHyperAlgebra* respectively, and we present several properties and examples connected with the real world.

Keywords: n-Power Set of a Set, n-SuperHyperGraph (n-SHG), n-SHG-vertex, n-SHG-edge, Plithogenic n-SuperHyperGraph, n-ary HyperOperation, n-ary HyperAxiom, n-ary HyperAlgebra, n-ary NeutroHyperOperation, n-ary NeutroHyperAxiom, n-ary NeutroHyperAlgebra, n-ary AntiHyperOperation, n-ary AntiHyperAxiom, n-ary AntiHyperAlgebra

1. Introduction

In this paper, with respect to the classical HyperGraph (that contains HyperEdges), we add the SuperVertices (a group of vertices put all together form a SuperVertex), in order to form a SuperHyperGraph (SHG). Therefore, each SHG-vertex and each SHG-edge belong to $P(V)$, where V is the set of vertices, and $P(V)$ means the power set of V .

Further on, since in our world we encounter complex and sophisticated groups of individuals and complex and sophisticated connections between them, we extend the SuperHyperGraph to n-SuperHyperGraph, by extending $P(V)$ to $P^n(V)$ that is the n-power set of the set V (see below). Therefore, the n-SuperHyperGraph, through its n-SHG-vertices and n-SHG-edges that belong to $P^n(V)$, can be the best (so far) to model our complex and sophisticated reality.

In the second part of the paper, we extend the classical HyperAlgebra to n-ary HyperAlgebra and its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra.

2. n-Power Set of a Set

Let U be a universe of discourse, and a subset $V \subseteq U$. Let $n \geq 1$ be an integer.

Let $P(V)$ be the *Power Set of the Set V* (i.e. all subsets of V , including the empty set \emptyset and the whole set V). This is the classical definition of power set.

For example, if $V = \{a, b\}$, then $P(V) = \{\emptyset, a, b, \{a, b\}\}$.

But we have extended the power set to *n-Power Set of a Set* [1].

For $n = 1$, one has the notation (identity): $P^1(V) \equiv P(V)$.

For $n = 2$, the 2-Power Set of the Set V is defined as follows:

$$P^2(V) = P(P(V)).$$

In our previous example, we get:

$$\begin{aligned} P^2(V) = P(P(V)) = P(\{\emptyset, a, b, \{a, b\}\}) = & \{\emptyset, a, b, \{a, b\}, \{\emptyset, a\}, \{\emptyset, b\}, \{\emptyset, \{a, b\}\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}; \\ & \{\emptyset, a, b\}, \{\emptyset, a, \{a, b\}\}, \{\emptyset, b, \{a, b\}\}, \{a, b, \{a, b\}\}; \{\emptyset, a, b, \{a, b\}\}\}. \end{aligned}$$

Definition of n-Power Set of a Set

In general, the **n-Power Set of a Set V** is defined as follows:

$$P^{n+1}(V) = P(P^n(V)), \text{ for integer } n \geq 1.$$

3. Definition of SuperHyperGraph (SHG)

A **SuperHyperGraph (SHG)** [1] is an ordered pair $SHG = (G \subseteq P(V), E \subseteq P(V))$, where

- (i) $V = \{V_1, V_2, \dots, V_m\}$ is a finite set of $m \geq 0$ vertices, or an infinite set.
- (ii) $P(V)$ is the power set of V (all subset of V). Therefore, an **SHG-vertex** may be a *single* (classical) vertex, or a super-vertex (a subset of many vertices) that represents a group (organization), or even an indeterminate-vertex (unclear, unknown vertex); \emptyset represents the null-vertex (vertex that has no element).
- (iii) $E = \{E_1, E_2, \dots, E_m\}$, for $m \geq 1$, is a family of subsets of V , and each E_i is an SHG-edge, $E_i \in P(V)$. An **SHG-edge** may be a (classical) edge, or a super-edge (edge between super-vertices) that represents connections between two groups (organizations), or hyper-super-edge that represents connections between three or more groups (organizations), multi-edge, or even indeterminate-edge (unclear, unknown edge); \emptyset represents the null-edge (edge that means there is no connection between the given vertices).

4. Characterization of the SuperHyperGraph

Therefore, a **SuperHyperGraph (SHG)** may have any of the below:

- *SingleVertices* (V_i), as in classical graphs, such as: V_1, V_2 , etc.;
- *SuperVertices (or SubsetVertices)* (SV_i), belonging to $P(V)$, for example: $SV_{1,3} = V_1V_3, SV_{2,57} = V_2V_{57}$, etc. that we introduce now for the first time. A super-vertex may represent a group (organization, team, club, city, country, etc.) of many individuals;

The comma between indexes distinguishes the single vertexes assembled together into a single SuperVertex. For example $SV_{12,3}$ means the single vertex S_{12} and single vertex S_3 are put together to form a super-vertex. But $SV_{1,2,3}$ means the single vertices S_1 and $S_{2,3}$ are put together; while $SV_{1,2,3}$ means S_1, S_2, S_3 as single vertices are put together as a super-vertex.

In no comma in between indexes, i.e. SV_{123} means just a single vertex V_{123} , whose index is 123, or $SV_{123} \equiv V_{123}$.

- *IndeterminateVertices* (i.e. unclear, unknown vertices); we denote them as: IV_1, IV_2 , etc. that we introduce now for the first time;
- *NullVertex* (i.e. vertex that has no elements, let's for example assume an abandoned house, whose all occupants left), denoted by $\emptyset V$.

- *SingleEdges*, as in classical graphs, i.e. edges connecting only two single-vertices, for example: $E_{1,5} = \{V_1, V_5\}$, $E_{2,3} = \{V_2, V_3\}$, etc.;
- *HyperEdges*, i.e. edges connecting three or more single-vertices, for example $HE_{1,4,6} = \{V_1, V_4, V_6\}$, $HE_{2,4,5,7,8,9} = \{V_2, V_4, V_5, V_7, V_8, V_9\}$, etc. as in hypergraphs;
- *SuperEdges* (or *SubsetEdges*), i.e. edges connecting only two SHG-vertices (and at least one vertex is SuperVertex), for example $SE_{(13,6),(45,79)} = \{SV_{13,6}, SV_{45,79}\}$ connecting two SuperVertices, $SE_{9,(2,345)} = \{V_9, SV_{2,345}\}$ connecting one SingleVertex V_9 with one SuperVertex, $SV_{2,345}$, etc. that we introduce now for the first time;
- *HyperSuperEdges* (or *HyperSubsetEdges*), i.e. edges connecting three or more vertices (and at least one vertex is SuperVertex, for example $HSE_{3,45,236} = \{V_3, V_{45}, V_{236}\}$, $HSE_{1234,456789,567,5679} = \{SV_{1234}, SV_{456789}, SV_{567}, SV_{5679}\}$, etc. that we introduce now for the first time;
- *MultiEdges*, i.e. two or more edges connecting the same (single-/super-/indeterminate-) vertices; each vertex is characterized by many attribute values, thus with respect to each attribute value there is an edge, the more attribute values the more edges (= multiedge) between the same vertices;
- *IndeterminateEdges* (i.e. unclear, unknown edges; either we do not know their value, or we do not know what vertices they might connect): IE_1, IE_2 , etc. that we introduce now for the first time;
- *NullEdge* (i.e. edge that represents no connection between some given vertices; for example two people that have no connections between them whatsoever): denoted by ϕE .

5. Definition of the n-SuperHyperGraph (*n-SHG*)

A **n-SuperHyperGraph** (*n-SHG*) [1] is an ordered pair $n-SHG = (G_n \subseteq P^n(V), E_n \subseteq P^n(V))$, where $P^n(V)$ is the *n*-power set of the set V , for integer $n \geq 1$.

6. Examples of 2-SuperHyperGraph, SuperVertex, IndeterminateVertex, SingleEdge, Indeterminate Edge, HyperEdge, SuperEdge, MultiEdge, 2-SuperHyperEdge

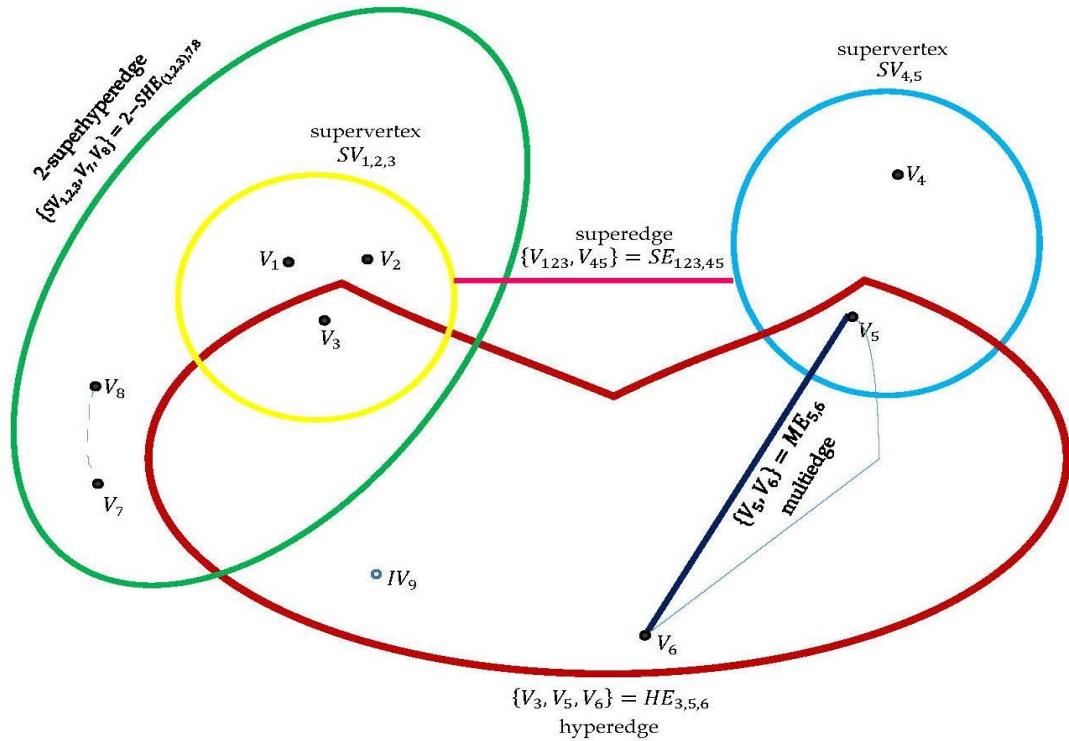


Figure 1. 2-SuperHyperGraph,
 $(IE_{7,8} = \text{Indeterminate Edge between single vertices } V_7 \text{ and } V_8, \text{ since the connecting curve is dotted,}$
 $IV_9 \text{ is an Indeterminate Vertex (since the dot is not filled in),}$
 $\text{while } ME_{5,6} \text{ is a MultiEdge (double edge in this case) between single vertices } V_5 \text{ and } V_6.$

Let V_1 and V_2 be two single-vertices, characterized by the attributes $a_1 = \text{size}$, whose attribute values are $\{\text{short, medium, long}\}$, and $a_2 = \text{color}$, whose attribute values are $\{\text{red, yellow}\}$. Thus we have the attributes values ($\text{Size}\{\text{short, medium, long}\}, \text{Color}\{\text{red, yellow}\}$), whence: $V_1(a_1\{s_1, m_1, l_1\}, a_2\{r_1, y_1\})$, where s_1 is the degree of short, m_1 degree of medium, l_1 degree of long, while r_1 is the degree of red and y_1 is the degree of yellow of the vertex V_1 .

And similarly $V_2(a_1\{s_2, m_2, l_2\}, a_2\{r_2, y_2\})$.

The degrees may be fuzzy, neutrosophic etc.

Example of fuzzy degree:

$V_1(a_1\{0.8, 0.2, 0.1\}, a_2\{0.3, 0.5\})$.

Example of neutrosophic degree:

$V_1(a_1\{(0.7, 0.3, 0.0), (0.4, 0.2, 0.1), (0.3, 0.1, 0.1)\}, a_2\{(0.5, 0.1, 0.3), (0.0, 0.2, 0.7)\})$.

Examples of the SVG-edges connecting single vertices V_1 and V_2 are below:

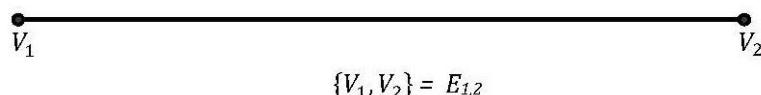


Figure 2. SingleEdge with respect to attributes a_1 and a_2 all together

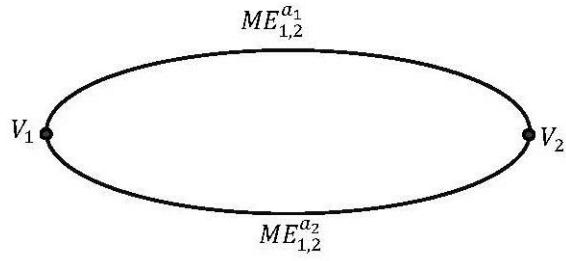


Figure 3. MultiEdge: top edge with respect to attribute a_1 , and bottom edge with respect to attribute a_2

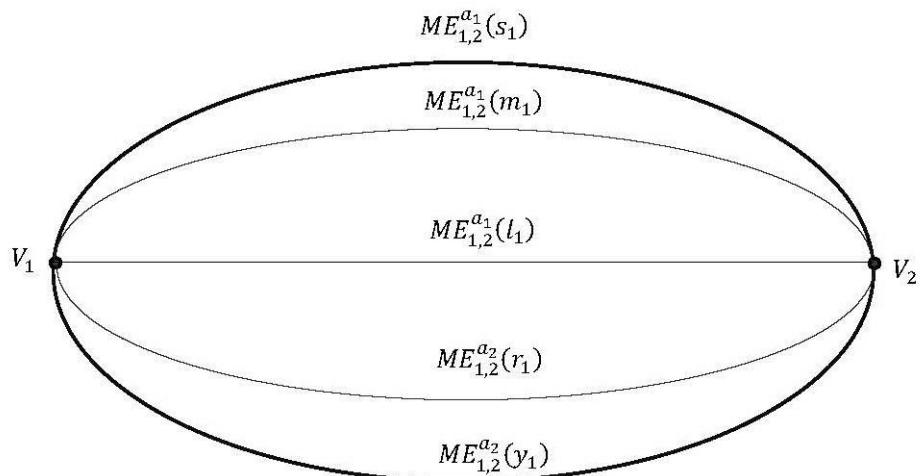


Figure 4. MultiEdge (= Refined MultiEdge from Figure 3):
the top edge from Figure 3, corresponding to the attribute a_1 , is split into three sub-edges with respect
to the attribute a_1 values s_1 , m_1 , and l_1 ;
while the bottom edge from Figure 3, corresponding to the attribute a_2 , is split into two sub-edges
with respect to the attribute a_2 values r_1 , and y_1 .

Depending on the application and on experts, one chooses amongst SingleEdge, MultiEdge, Refined-MultiEdge, RefinedRefinedMultiEdge, etc.

7. Plithogenic n-SuperHyperGraph

As a consequence, we introduce for the first time the Plithogenic n-SuperHyperGraph.

A **Plithogenic n-SuperHyperGraph (n-PSHG)** is a n-SuperHyperGraph whose each *n-SHG-vertex* and each *n-SHG-edge* are characterized by many distinct attributes values ($a_1, a_2, \dots, a_p, p \geq 1$).

Therefore one gets *n-SHG-vertex*(a_1, a_2, \dots, a_p) and *n-SHG-edge*(a_1, a_2, \dots, a_p).

The attributes values degrees of appurtenance to the graph may be crisp / fuzzy / intuitionistic fuzzy / picture fuzzy / spherical fuzzy / etc. / neutrosophic / refined neutrosophic / degrees with respect to each *n-SHG-vertex* and each *n-SHG-edge* respectively.

For example, one has:

Fuzzy-*n-SHG-vertex*($a_1(t_1), a_2(t_2), \dots, a_p(t_p)$) and Fuzzy-*n-SHG-edge*($a_1(t_1), a_2(t_2), \dots, a_p(t_p)$);

Intuitionistic Fuzzy-*n-SHG-vertex*($a_1(t_1, f_1), a_2(t_2, f_2), \dots, a_p(t_p, f_p)$)

and Intuitionistic Fuzzy- n -SHG-edge($a_1(t_1, f_1), a_2(t_2, f_2), \dots, a_p(t_p, f_p)$);
 Neutrosophic- n -SHG-vertex($a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \dots, a_p(t_p, i_p, f_p)$)
 and Neutrosophic- n -SHG-edge($a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \dots, a_p(t_p, i_p, f_p)$);
 etc.

Whence we get:

8. The Plithogenic (Crisp / Fuzzy / Intuitionistic Fuzzy / Picture Fuzzy / Spherical Fuzzy / etc. / Neutrosophic / Refined Neutrosophic) n -SuperHyperGraph.

9. Introduction to n -ary HyperAlgebra

Let U be a universe of discourse, a nonempty set $S \subset U$. Let $P(S)$ be the power set of S (i.e. all subsets of S , including the empty set ϕ and the whole set S), and an integer $n \geq 1$.

We formed [2] the following neutrosophic triplets, which are defined in below sections:

(n -ary HyperOperation, n -ary NeutroHyperOperation, n -ary AntiHyperOperation),

(n -ary HyperAxiom, n -ary NeutroHyperAxiom, n -ary AntiHyperAxiom), and

(n -ary HyperAlgebra, n -ary NeutroHyperAlgebra, n -ary AntiHyperAlgebra).

10. n -ary HyperOperation (n -ary HyperLaw)

A n -ary HyperOperation (n -ary HyperLaw) $*_n$ is defined as:

$*_n : S^n \rightarrow P(S)$, and

$\forall a_1, a_2, \dots, a_n \in S$ one has $*_n(a_1, a_2, \dots, a_n) \in P(S)$.

The n -ary HyperOperation (n -ary HyperLaw) is well-defined.

11. n -ary HyperAxiom

A n -ary HyperAxiom is an axiom defined of S , with respect the above n -ary operation $*_n$, that is true for all n -plets of S^n .

12. n -ary HyperAlgebra

A n -ary HyperAlgebra ($S, *_n$), is the S endowed with the above n -ary well-defined HyperOperation $*_n$.

13. Types of n -ary HyperAlgebras

Adding one or more n -ary HyperAxioms to S we get different types of n -ary HyperAlgebras.

14. n -ary NeutroHyperOperation (n -ary NeutroHyperLaw)

A n -ary NeutroHyperOperation is a n -ary HyperOperation $*_n$ that is well-defined for some n -plets of S^n

[i.e. $\exists(a_1, a_2, \dots, a_n) \in S^n, *_n(a_1, a_2, \dots, a_n) \in P(S)$],

and indeterminate [i.e. $\exists(b_1, b_2, \dots, b_n) \in S^n, *_n(b_1, b_2, \dots, b_n) = \text{indeterminate}$]

or outer-defined [i.e. $\exists(c_1, c_2, \dots, c_n) \in S^n, *_n(c_1, c_2, \dots, c_n) \notin P(S)$] (or both), on other n -plets of S^n .

15. n-ary NeutroHyperAxiom

A n -ary *NeutroHyperAxiom* is an n -ary HyperAxiom defined of S , with respect the above n -ary operation $*_n$, that is true for some n -plets of S^n , and indeterminate or false (or both) for other n -plets of S^n .

16. n-ary NeutroHyperAlgebra is an n -ary HyperAlgebra that has some n -ary NeutroHyperOperations or some n -ary NeutroHyperAxioms

17. n-ary AntiHyperOperation (n-ary AntiHyperLaw)

A n -ary *AntiHyperOperation* is a n -ary HyperOperation $*_n$ that is outer-defined for all n -plets of S^n [i.e.

$$\forall(s_1, s_2, \dots, s_n) \in S^n, *_n(s_1, s_2, \dots, s_n) \notin P(S).$$

18. n-ary AntiHyperAxiom

A n -ary *AntiHyperAxiom* is an n -ary HyperAxiom defined of S , with respect the above n -ary operation $*_n$ that is false for all n -plets of S^n .

19. n-ary AntiHyperAlgebra is an n -ary HyperAlgebra that has some n -ary AntiHyperOperations or some n -ary AntiHyperAxioms.

20. Conclusion

We have recalled our 2019 concepts of n -Power Set of a Set, n -SuperHyperGraph and Plithogenic n -SuperHyperGraph [1], afterwards the n -ary HyperAlgebra together with its alternatives n -ary NeutroHyperAlgebra and n -ary AntiHyperAlgebra [2], and we presented several properties, explanations, and examples inspired from the real world.

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